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Estimates of Covering Numbers

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For operators between Banach spaces, we study certain s-covering numbers, which are a kind of combination between s-numbers and entropy numbers. We prove inequalities between s-covering numbers and various s-numbers. As an application, minoration theorems involving the ℓ-norm and γ-norm are given.

1. INTRODUCTION

In this paper we establish inequalities involving s-numbers and so-called s-covering numbers of operators between Banach spaces. One of the main results, which is proved in Section 3 (Theorem 3.4), is the following weak type inequality, in which the a-covering numbers of operators are dominated by the approximation numbers

$$\sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{n,l}(T) \leqslant C(p) \sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{l+n-1}(T), \tag{1.1}$$

for $0 < p < \infty$, $1 \leqslant l, m < \infty$. Various inequalities of weak type, used to majorize other s-covering numbers by s-numbers, can be verified with the help of this inequality. In order to prove (1.1) we refer to a striking result of Pisier [P1, P2] concerning the existence of isomorphisms from arbitrary n-dimensional Banach spaces into the n-dimensional Hilbert space such

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that certain s-numbers of these isomorphisms can be appropriately majorized.

A number of useful applications of (1.1) are given in Section 4. We derive so-called minoration theorems for Gaussian averages (l-norms) and Rademacher averages (r-norms) of operators from ℓ_2 into Banach spaces. The results generalize the classical minoration theorem of Sudakov for the ℓ-norm, which plays an important role in the theory of stochastic processes. The estimates can be considered as a kind of interpolation or combination between Sudakov's minoration theorem and a minoration theorem of the ℓ-norm, which originates in the paper of V. Milman [M1] and which was improved by A. Pajor and N. Tomczak-Jaegermann [PaT].

Throughout this paper we use standard definitions and notations of Banach space theory. For the sake of convenience we recall some of them. Throughout this paper E, F, and G denote (real or complex) Banach spaces. Given a Banach space E we denote its closed unit ball by U_E and its dual space by E'. Moreover $L(E, F)$ denotes the Banach space of all (bounded linear) operators from E into F equipped with the usual operator norm.

In order to describe the covering concept let us recall the definition of the dyadic entropy numbers $e_n(T)$. The nth dyadic entropy number $e_n(T)$, $n \geqslant 1$, of an operator $T \in L(E, F)$ is defined to be the infimum of all $\varepsilon > 0$ such that there exist $y_1, ..., y_{2^{n-1}} \in F$ for which

$$T(U_E) \subseteq \bigcup_{i=1}^{2^{n-1}} \{y_i + \varepsilon U_F\}$$

holds. For the definition of s-numbers we refer to [Pi] or [CS].

2. s-Covering Numbers

According to the concepts of s-numbers and dyadic entropy numbers we define an s-covering number function. Let L denote the class of all (bounded linear) operators, $\mathbb{N}$ the set of natural numbers, and $s: L \to \ell_\infty(\mathbb{N})$ any s-number function. An s-covering number function is a map

$$s = (s_{n,k}): L \to \ell_\infty(\mathbb{N} \times \mathbb{N})$$

which associates with every operator T its s-covering numbers $s_{n,k}(T)$ and satisfies the following properties:

(SC i) *Combination Property.*

$$s_{n,1}(T) = e_n(T), \qquad s_{1,n}(T) = s_n(T) \qquad \text{for} \quad T \in L(E, F) \text{ and } 1 \leqslant n < \infty.$$

(SC ii) *Monotonicity.*

$$\|T\| = s_{1,1}(T) \geqslant s_{m,j}(T) \geqslant s_{n,k}(T) \geqslant 0$$
$$\text{for} \quad T \in L(E, F) \text{ and } n \geqslant m \geqslant 1 \text{ and } k \geqslant j \geqslant 1.$$

(SC iii) *Additivity.*

$$s_{m+n-1, j+k-1}(S+T) \leqslant s_{m,j}(S) + s_{n,k}(T)$$
$$\text{for} \quad S, T \in L(E, F) \text{ and } 1 \leqslant j, k, m, n < \infty.$$

(SC iv) *Ideal Property.*

$$s_{n,k}(RST) \leqslant \|R\| \, s_{n,k}(S) \, \|T\|$$
$$\text{for} \quad R \in L(G, H), S \in L(F, G), T \in L(E, F), \text{ and } 1 \leqslant k, n < \infty.$$

Moreover, some s-covering number functions satisfy an additional property called

(SC v) *C-Multiplicativity.* There exists a constant $C \geqslant 1$ such that

$$s_{m+n-1, j+k-1}(ST) \leqslant C \cdot s_{m,j}(S) \, s_{n,k}(T)$$
$$\text{for} \quad S \in L(F, G), T \in L(E, F), \text{ and } 1 \leqslant j, k, m, n < \infty.$$

If $C = 1$, then the s-covering number function is said to be multiplicative. This concept was introduced in [CM] in a slightly modified version.

The basic examples are the

a-Covering numbers.

$$a_{n,k}(T) = \inf\{e_n(T - A) \colon \operatorname{rank}(A) < k\}. \tag{2.1}$$

This means that for $T \in L(E, F)$, $a_{n,k}(T)$ is the infimum of all $\rho > 0$ such that there exists an operator $A \in L(E, F)$ with $\operatorname{rank}(A) < k$ and elements $y_i \in F$, $1 \leqslant i \leqslant 2^{n-1}$ for which

$$(T - A)(U_E) \subseteq \bigcup_{i=1}^{2^{n-1}} \{y_i + \rho U_F\}.$$

Analogously:

c-Covering numbers.

$$c_{n,k}(T) = \inf\{e_n(TI_M^E) \colon M \subseteq E, \operatorname{codim}(M) < k\}. \tag{2.2}$$

d-Covering numbers.

$$d_{n,k}(T)=\inf\{e_n(Q_N^F T)\colon N\subseteq F,\ \dim(N)<k\}. \tag{2.3}$$

t-Covering numbers.

$$t_{n,k}(T)=a_{n,k}(I_\infty TQ_1), \tag{2.4}$$

where $Q_1\colon \ell_1(U_E)\to E$ and $I_\infty\colon F\to \ell_\infty(U_{F'})$ are the canonical surjection and injection, respectively. (For the t-covering numbers we have a weaker form of the combination property, namely

$$\tfrac{1}{2}e_n(T)\leqslant t_{n,1}(T)\leqslant e_n(T).)$$

These s-covering number functions were already considered in [CM] together with the so-called

e-Covering numbers.

$$e_{n,k}(T)=\inf\{\varepsilon>0\colon \text{there exist subspaces } M_i\subseteq F,\ \dim(M_i)<k, \text{ and } y_i\in F,\ 1\leqslant i\leqslant 2^{n-1}, \text{ such that } T(U_E)\subseteq \textstyle\bigcup_{i=1}^{2^{n-1}}\{y_i+M_i+\varepsilon U_F\}\}. \tag{2.5}$$

Note that the combination property (SC i) coincides for the e- and d-covering numbers since $e_{n,1}(T)=e_n(T)$ and $e_{1,n}(T)=d_n(T)$. The difference between (2.3) and (2.5) becomes clear if we rewrite (2.3) into

$$d_{n,k}(T)=\inf\{\varepsilon>0\colon \text{there exists a subspace } M\subseteq F,\ \dim(M)<k, \text{ and } y_i\in F,\ 1\leqslant i\leqslant 2^{n-1}, \text{ such that } T(U_E)\subseteq \textstyle\bigcup_{i=1}^{2^{n-1}}\{y_i+M_i+\varepsilon U_F\}\},$$

which means that in the definition of $d_{n,k}$ the coverings consist of "cylinders" $\{y_i+M_i+\varepsilon U_F\}$ with one common "direction" M while the directions may be different in (2.5). From this we obviously obtain

$$e_{n,k}(T)\leqslant d_{n,k}(T).$$

3. Inequalities between s-Covering Numbers and s-Numbers

This section deals with basic estimates between s-covering numbers and s-numbers.

LEMMA 3.1. *Let $A\in L(E,F)$ be an operator of finite rank,* $\operatorname{rank}(A)=r$, *acting between real Banach spaces E and F. Furthermore, let $N=N(A)$ and*

$R = R(A)$ be the null space and range of A, respectively. For arbitrary isomorphisms

$$X\colon E/N \to \ell_2^r \qquad \text{and} \qquad Y\colon R \to \ell_2^r$$

we have

$$a_{3n-2,l}(A) \leqslant 7 e_n(X)\, e_n(Y^{-1}) \sup_{1 \leqslant k \leqslant M} \Bigg[2^{-n/(3k+3)} \times \left\{ \prod_{i=1}^{k} c_i(Y)\, d_i(X^{-1}) \right\}^{1/k} \left\{ \sum_{i=1}^{k} t_{l+i-1}(A) \right\}^{1/k} \Bigg], \tag{3.1}$$

for $1 \leqslant l, n < \infty$ with

$$M = \max\left\{1, \left[1 + \frac{r-l}{3}\right]\right\}, \tag{3.2}$$

where $[x]$ denotes the integer part of x.

Proof. Without loss of generality we may assume that

$$r = \operatorname{rank}(A) \geqslant 1, \tag{3.3}$$

since $a_{3n-2,l}(A) \leqslant a_{1,l}(A) = a_1(A) = 0$ in the contrary case. We factorize A canonically through the quotient map $Q\colon E \to E/N$ and the imbedding $I_R^F\colon R \to F$,

$$A = I_R^F A_0 Q, \tag{3.4}$$

and use the isomorphisms $X\colon E/N \to \ell_2^r$ to introduce $S\colon \ell_2^r \to \ell_2^r$ by

$$S = Y A_0 X^{-1} \qquad \text{or, equivalently,} \qquad A_0 = Y^{-1} S X. \tag{3.5}$$

Next we estimate the a-covering numbers of S. According to the Schmidt representation formula there exist isometries $U, V\colon \ell_2^r \to \ell_2^r$ and a diagonal operator D with positive entries such that

$$S = U D V^{-1} \qquad \text{and} \qquad D = U^{-1} S V. \tag{3.6}$$

If the diagonal elements σ_i of D are ordered in a non-increasing sequence we then can express σ_i by

$$\sigma_i = a_i(S) = c_i(S) = d_i(S) = t_i(S), \qquad 1 \leqslant i \leqslant r. \tag{3.7}$$

The ideal property (SC iv) and (3.6) imply that

$$a_{n,l}(S) = a_{n,l}(D), \qquad 1 \leqslant l, n < \infty. \tag{3.8}$$

An estimate for the $a_{n,l}(D)$ can be obtained as follows. Consider D_l: $\ell_2^r \to \ell_2^r$, $D_l(\xi_1, \xi_2, ..., \xi_r) = (\sigma_1 \xi_1, ..., \sigma_{l-1} \xi_{l-1}, 0, ..., 0)$. According to the definition of $a_{n,l}$ and a result of Gordon, König, and Schütt (cf. [CS, GKS]) we obtain

$$a_{n,l}(D) \leqslant e_n(D - D_l) \leqslant 6 \sup_{1 \leqslant k \leqslant r-l+1} 2^{-(n-1)/k} \left\{ \prod_{i=1}^{k} \sigma_{l+i-1} \right\}^{1/k}. \tag{3.9}$$

It follows from (3.7) and (3.8) that

$$a_{n,l}(S) \leqslant 6 \sup_{1 \leqslant k \leqslant r-l+1} 2^{-(n-1)/k} \left\{ \prod_{i=1}^{k} t_{l+i-1}(S) \right\}^{1/k}. \tag{3.10}$$

Note that the range of k is not empty because of (3.3).

In order to replace the operator S by A we use (3.5) and the multiplicativity, injectivity, and surjectivity of the symmetrized approximation numbers t_i. The multiplicativity of the t_i allows us to write

$$t_{i+j+k-2}(RST) \leqslant c_i(R)\, t_j(S)\, d_k(T)$$

(cf. [CS]). Hence we obtain from (3.5)

$$t_{l+3i-3}(S) \leqslant c_i(Y)\, t_{l+i-1}(A_0)\, d_i(X^{-1}) = c_i(Y)\, t_{l+i-1}(A)\, d_i(X^{-1}). \tag{3.11}$$

Now we insert (3.11) into (3.10). For this purpose let

$$\gamma_j = \left\{ \prod_{i=1}^{j} t_{l+i-1}(S) \right\}^{1/j}$$

be the non-increasing sequence of geometric means of $t_l(S), t_{l+1}(S), ...$. Then

$$2^{-(n-1)/j}\gamma_j \leqslant 2^{-(n-1)/6}\gamma_3 \leqslant 2^{-(n-1)/6} t_l(S) \qquad \text{for} \quad j = 4, 5, 6;$$

$$2^{-(n-1)/j}\gamma_j \leqslant 2^{-(n-1)/(3k+3)}\gamma_{3k} \leqslant 2^{-(n-1)/(3k+3)} \left\{ \prod_{i=1}^{k} t_{l+3i-3}(S) \right\}^{1/k}$$

$$\text{for} \quad j = 3k+1, 3k+2, 3k+3.$$

Furthermore

$$2^{-(n-1)/j}\gamma_j \leqslant 2^{-(n-1)/6} t_l(S) \qquad \text{for} \quad j = 1, 2, 3.$$

Hence (3.10) becomes

$$\begin{aligned} a_{n,l}(S) &\leqslant 6 \sup_{1\leqslant k<\infty} 2^{-(n-1)/(3k+3)} \left\{\prod_{i=1}^{k} t_{l+3i-3}(S)\right\}^{1/k} \\ &\leqslant 7 \sup_{1\leqslant k<\infty} 2^{-n/(3k+3)} \left\{\prod_{i=1}^{k} t_{l+3i-3}(S)\right\}^{1/k}, \end{aligned} \tag{3.12}$$

where non-zero terms occur only for $l+3k-3\leqslant r$, i.e., $1\leqslant k\leqslant M$, cf. (3.2). From (3.11) and (3.12) we obtain

$$a_{n,l}(S)\leqslant 7 \sup_{1\leqslant k\leqslant M} 2^{-n/(3k+3)} \left\{\prod_{i=1}^{k} c_i(Y)\, d_i(X^{-1})\right\}^{1/k} \left\{\prod_{i=1}^{k} t_{l+i-1}(A)\right\}^{1/k}.$$

The multiplicativity (SC v) and (3.4), (3.5) imply that

$$a_{3n-2,l}(A)\leqslant a_{3n-2,l}(A_0)\leqslant e_n(Y^{-1})\, a_{n,l}(S)\, e_n(X);$$

therefore

$$\begin{aligned} a_{3n-2,l}(A)\leqslant 7e_n(X)\, e_n(Y^{-1}) \sup_{1\leqslant k\leqslant M} \Bigg[& 2^{-n/(3k+3)} \\ & \times\left\{\prod_{i=1}^{k} c_i(Y)\, d_i(X^{-1})\right\}^{1/k} \left\{\prod_{i=1}^{k} t_{l+i-1}(A)\right\}^{1/k}\Bigg]. \end{aligned}$$ ∎

Remark 3.1. If E and F are complex Banach spaces we must replace

$$2^{-n/(3k+3)} \text{ by } 2^{-n/(6k+6)} \text{ in (3.1)}.$$

From this lemma we derive some conclusions. Let us first recall the definition of local injective, resp. surjective distances, based on the Banach–Mazur distance $d(E, F)$ of Banach spaces E, F:

$$d(E,F)=\begin{cases}\inf\{\|T\|\,\|T^{-1}\| \text{ for all isomorphisms } T\in L(E,F)\}\\ \infty \quad \text{if } E \text{ and } F \text{ are not isomorphic.}\end{cases}$$

The nth local injective distance $\delta_n(E)$ is given as

$$\delta_n(E)=\sup\{d(M,\ell_2^m)\colon M\subseteq E,\, m=\dim(M)\leqslant n\},$$

and the nth local surjective distance $\delta^{(n)}(E)$ as

$$\delta^{(n)}(E)=\sup\{d(E/M,\ell_2^m)\colon M\subseteq E,\, m=\operatorname{codim}(M)\leqslant n\}.$$

COROLLARY 3.1. *Under the assumptions of Lemma* 3.1 *the following inequality holds*

$$a_{n,l}(A)\leqslant 7\delta^{(r)}(E)\,\delta_r(F) \sup_{1\leqslant k\leqslant M} 2^{-n/(9k+9)} \left\{\prod_{i=1}^{k} t_{l+i-1}(A)\right\}^{1/k}. \tag{3.13}$$

Proof. Given $\varepsilon > 0$ we choose X and Y in Lemma 3.1 with

$$\|X\| \, \|X^{-1}\| \leqslant (1+\varepsilon)\, d(E/N, \ell_2^r) \leqslant (1+\varepsilon)\, \delta^{(r)}(E)$$

$$\|Y\| \, \|Y^{-1}\| \leqslant (1+\varepsilon)\, d(R, \ell_2^r) \leqslant (1+\varepsilon)\, \delta_r(F).$$

All terms in (3.1) containing X or Y are estimated by the operator norms. Hence for $3m-2 \leqslant n \leqslant 3m$ we have

$$a_{n,l}(A) \leqslant a_{3m-2,l}(A) \leqslant 7(1+\varepsilon)^2\, \delta^{(r)}(E)\, \delta_r(F) \times \sup_{1 \leqslant k \leqslant M} 2^{-m/(3k+3)} \left\{ \prod_{i=1}^{k} t_{l+i-1}(A) \right\}^{1/k}.$$

Since $m \geqslant n/3$ the proof is complete.

Now an important result of Pisier [P1, P2] is used to obtain further consequences of Lemma 3.1.

THEOREM 3.1. (Pisier). *For each $\alpha > \frac{1}{2}$ there is a constant $C(\alpha)$ such that for any n-dimensional (real or complex) Banach space E there is an isomorphism X from E onto ℓ_2^n (real or complex, respectively), such that*

$$d_k(X) \leqslant C(\alpha) \left(\frac{n}{k}\right)^{\alpha} \quad \text{and} \quad d_k(X^{-1}) \leqslant C(\alpha) \left(\frac{n}{k}\right)^{\alpha} \tag{3.14}$$

for all k, $1 \leqslant k \leqslant n$. (For $k > n$ we have $d_k(X) = c_k(X^{-1}) = 0$ in any case.) Moreover, the constant $C(\alpha)$, only depending on α, can be chosen of order $O(\alpha - \frac{1}{2})^{-1}$ for $\alpha \downarrow \frac{1}{2}$. ∎

The corresponding result concerning the dyadic entropy numbers of X follows immediately from Theorem 3.1 and the following inequality from [C] (cf. [CS, P2].

THEOREM 3.2. *For each p, $p > 0$, there is a constant c_p such that for all operators T we have*

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(T) \leqslant c_p \sup_{1 \leqslant k \leqslant n} k^{1/p} t_k(T), \qquad 1 \leqslant n < \infty, \tag{3.15}$$

and c_p remains bounded if p varies in a compact subset of $(0, \infty)$.

If we note that $t_k(X) = d_k(I_\infty X) \leqslant d_k(X)$ and analogously $t_k(X^{-1}) = c_k(X^{-1} Q_1) \leqslant c_k(X^{-1})$ we obtain from (3.14) and (3.15), by setting $p = 1/\alpha$,

$$k^\alpha e_k(T) \leqslant c_{1/\alpha} \sup_{1 \leqslant k \leqslant n} k^\alpha t_k(T) \leqslant c_{1/\alpha} \sup_{1 \leqslant k \leqslant n} k^\alpha C(\alpha) \left(\frac{n}{k}\right)^{\alpha}$$

$$= c_{1/\alpha} C(\alpha)\, n^\alpha, \qquad 1 \leqslant k \leqslant n.$$

Hence we may state the following corollary.

COROLLARY 3.2. *Let $X: E \to \ell_2^n$ be the isomorphism from Theorem* 3.1. *Then*

$$\max\{e_k(X), e_k(X^{-1})\} \leqslant c_{1,\alpha} C(\alpha) \left(\frac{n}{k}\right)^{\alpha} \quad \textit{for all} \quad k, 1 \leqslant k < \infty, \tag{3.16}$$

where $C(\alpha)$ and $c_{1,\alpha}$ are the constants of (3.14) *and* (3.15). *Moreover, for a modified $C(\alpha)$, which is also of order $O(\alpha - \frac{1}{2})^{-1/2}$ for $\alpha \downarrow \frac{1}{2}$, we have*

$$\max\{e_k(X), e_k(X^{-1}), d_k(X), c_k(X^{-1})\} \leqslant C(\alpha) \left(\frac{n}{k}\right)^{\alpha} \quad \textit{for} \quad 1 \leqslant k < \infty. \tag{3.17}$$

These considerations lead to a more sophisticated corollary of Lemma 3.1.

COROLLARY 3.3. *For any β, $\beta > 1$, there exists a constant $C(\beta)$ of order $C(\beta) = O(\beta - 1)^{-2}$ for $\beta \downarrow 1$ such that for any operator $A \in L(E, F)$ with* $\operatorname{rank}(A) = r$ *between real Banach spaces E and F*

$$a_{n,l}(A) \leqslant C(\beta) \left(\frac{r}{n}\right)^{2\beta} \sup_{1 \leqslant k \leqslant M} 2^{-n/(9k+9)} \left(\frac{n}{9k+9}\right)^{\beta} \left\{\prod_{i=1}^{k} t_{l+i-1}(A)\right\}^{1/k}, \tag{3.18}$$

for all l, n, $1 \leqslant l, n < \infty$, M being given by (3.2).

Proof. Put $\alpha = \beta/2 > \frac{1}{2}$. Then by 3.17 there are X and Y such that

$$\max\{e_k(X), e_k(Y^{-1}), c_k(Y), d_k(X^{-1})\} \leqslant C(\alpha) \left(\frac{r}{k}\right)^{\alpha}, \qquad 1 \leqslant k < \infty. \tag{3.19}$$

We know that $C(\alpha) = O(\alpha - \frac{1}{2})^{-1/2} = O(\beta - 1)^{-1/2}$ for $\beta \downarrow 1$. Hence it follows from (3.1) that

$$a_{3n-2,l}(A) \leqslant 7e^2 C^4(\alpha) \left(\frac{r}{n}\right)^{\beta} \sup_{1 \leqslant k \leqslant M} 2^{-n/(3k+3)} \left(\frac{r}{k}\right)^{\beta} \left\{\prod_{i=1}^{k} t_{l+i-1}(A)\right\}^{1/k},$$

where $k! \geqslant (k/e)^k$ is used. Estimating

$$\left(\frac{1}{k}\right)^{\beta} \leqslant 18^{\beta} \left(\frac{n}{9k+9}\right)^{\beta}$$

and passing from $a_{3n-2,l}(A)$ to $a_{n,l}(A)$ as in the proof of Corollary 3.1, we finally obtain the inequality (3.18) with the constant $C(\beta)=7e^2 18^\beta C^4(\alpha)$ which is indeed of order $O(\beta-1)^{-2}$ for $\beta\downarrow 1$. ▮

Remark 3.2. If the Banach spaces X and Y are complex Banach spaces then $2^{-n/(9k+9)}$ in (3.13) and (3.18) must be replaced by $2^{-n/(18k+18)}$.

Now let us remove the restriction on A being a finite rank operator.

PROPOSITION 3.1. *Let $\beta>1$ and $T\in L(E,F)$ be an operator between real Banach spaces. Then*

$$a_{n,l}(T)\leqslant a_{3m+l}(T)+D(\beta)\left(\frac{m+l}{n}\right)^{2\beta}\sup_{1\leqslant k\leqslant m}\left[2^{-n/(9k+9)}\times\left(\frac{n}{9k+9}\right)^{\beta}\left\{\prod_{i=1}^{k}a_{l+i-1}(T)\right\}^{1/k}\right]\tag{3.20}$$

for $1\leqslant l,m,n<\infty$ with a constant $D(\beta)$, depending only on β and being of order $O(\beta-1)^{-2}$ for $\beta\downarrow 1$.

Proof. Given $\varepsilon>0$ we determine an operator $A\in L(E,F)$ with $r=\operatorname{rank}(A)<3m+1$ such that

$$\|T-A\|\leqslant(1+\varepsilon)\,a_{3m+l}(T).\tag{3.21}$$

From

$$a_{n,l}(T)\leqslant\|T-A\|+a_{n,l}(A)$$

(cf. (SC iii) and by applying (3.18) to A and using $r<3(m+l)$ and $M=\max\{1,[1+(r-l)/3]\}\leqslant m$ we obtain

$$a_{n,l}(T)\leqslant(1+\varepsilon)\,a_{3m+l}(T)+9^\beta C(\beta)\left(\frac{m+l}{n}\right)^{2\beta}\sup_{1\leqslant k\leqslant m}\left[2^{-n/(9k+9)}\times\left(\frac{n}{9k+9}\right)^{\beta}\left\{\prod_{i=1}^{k}t_{l+i-1}(A)\right\}^{1/k}\right].\tag{3.22}$$

In order to eliminate $t_{l+i-1}(A)$ we derive from (3.21)

$$\begin{aligned}t_{l+i-1}(A)&\leqslant\|A-T\|+t_{l+i-1}(T)\leqslant(1+\varepsilon)\,a_{3m+l}(T)+a_{l+i-1}(T)\\&\leqslant(2+\varepsilon)\,a_{l+i-1}(T)\qquad\text{for}\quad 1\leqslant i\leqslant m.\end{aligned}$$

Hence, since for $\beta\downarrow 1$

$$D(\beta)=2\cdot 9^\beta C(\beta)=O(\beta-1)^{-2}$$

(3.22) implies the desired estimate (3.20). ▮

The next inequality relates $a_{n,l}(T)$ to the $\ell_{p,\infty}$-norm of a finite subsequence of the approximation number sequence.

PROPOSITION 3.2. *Let $0<p<\infty$ and $T\in L(E,F)$ (E,F real Banach spaces). Then*

$$a_{n,l}(T)\leqslant a_{l+m-1}(T)+C(p)\left(\frac{m+l}{n}\right)^2\log^2\left(\frac{m+l}{n}+2\right)$$

$$\times n^{-1/p}\sup_{1\leqslant k\leqslant m} k^{1/p}a_{l+k-1}(T), \tag{3.23}$$

for all l, m, n, $1\leqslant l, m, n<\infty$, where $C(p)$ is a constant depending only on p.

Proof. Because of $k!\geqslant (k/e)^k$ we have

$$\left\{\prod_{i=1}^{k} a_{l+i-1}(T)\right\}^{1/k}$$

$$\leqslant (k!)^{-1/pk}\sup_{1\leqslant j\leqslant k} j^{1/p}a_{l+j-1}(T)$$

$$\leqslant\left(\frac{e}{k}\right)^{1/p}\sup_{1\leqslant k\leqslant m} k^{1/p}a_{l+k-1}(T)\qquad\text{for}\quad 1\leqslant k\leqslant m. \tag{3.24}$$

We now insert (3.24) into (3.20) and choose an appropriate $\beta>1$. Note that for the first summand on the right-hand side of (3.20)

$$a_{l+m-1}(T)\geqslant a_{3m+l}(T).$$

For the second summand we apply (3.24) and get from (3.20)

$$a_{n,l}(T)\leqslant a_{l+m-1}(T)+e^{1/p}D(\beta)\left(\frac{m+l}{n}\right)^{2\beta}\left\{\sup_{1\leqslant k\leqslant m} k^{1/p}a_{l+k-1}(T)\right\}$$

$$\times\left\{\sup_{1\leqslant k\leqslant m} 2^{-n/(9k+9)}\left(\frac{n}{9k+9}\right)^{\beta}k^{-1/p}\right\}. \tag{3.25}$$

Estimating $k^{-1/p}$ by

$$k^{-1/p}\leqslant 18^{1/p}n^{-1/p}\left(\frac{n}{9k+9}\right)^{1/p}$$

and setting $x=n/(9k+9)$, we may rewrite (3.25) as

$$a_{n,l}(T)\leqslant a_{l+m-1}(T)+(18e)^{1/p}D(\beta)\left(\frac{m+l}{n}\right)^{2\beta}n^{-1/p}$$

$$\times\left\{\sup_{1\leqslant k\leqslant m} k^{1/p}a_{l+k-1}(T)\right\}\left\{\sup_{0<x<\infty} 2^{-x}x^{\beta+1/p}\right\}. \tag{3.26}$$

With the function $\sigma(t) := \sup_{0<x<\infty} 2^{-x}x^t = (t/e \cdot \ln 2)^t$ and the constant $C(\beta, p) := (18e)^{1/p} D(\beta)\,\sigma(\beta + 1/p)$ we get from (3.26)

$$a_{n,l}(T) \leqslant a_{l+m-1}(T) + C(\beta, p)\left(\frac{m+l}{n}\right)^{2(\beta-1)}\left(\frac{m+l}{n}\right)^2 n^{-1/p} \times \{\sup_{1\leqslant k\leqslant m} k^{1/p} a_{l+k-1}(T)\}. \tag{3.27}$$

Fixing p, we see from the definition of $C(\beta, p)$ that $C(\beta, p) = O(\beta - 1)^{-2}$ for $\beta \downarrow 1$. If we set

$$\beta = 1 + \left[\log\left(\frac{m+l}{n} + 2\right)\right]^{-1} \tag{3.28}$$

then $1 < \beta < 3$ for all $l, m, n \geqslant 1$ and thus there is a constant $C(p)$ such that

$$C(\beta, p) \leqslant C(p)(\beta - 1)^{-2} \qquad \text{for} \quad 1 < \beta < 3.$$

With β given in (3.28) it is easy to verify that

$$C(\beta, p)\left(\frac{m+n}{n}\right)^{2(\beta-1)} \leqslant e^2 C(p)\left[\log\left(\frac{m+l}{n} + 2\right)\right]^2.$$

Together with (3.27) this completes the proof. ∎

Remark 3.3. In case of complex Banach spaces the value $2^{-n/(9k+9)}$ must be replaced by $2^{-n/(18k+8)}$ in (3.20), whereas (3.23) holds in the real and complex cases (clearly $C(p)$ must be modified).

The next theorems deal with inequalities related to Theorem 3.2.

THEOREM 3.3. *Let $0 < p < \infty$ and $T \in L(E, F)$, where E and F can be either real or complex Banach spaces. Then there exists a constant $C(p)$, only depending on p, such that*

$$\sup_{1\leqslant n\leqslant m} n^{1/p} a_{n,l}(T) \leqslant C(p) \sup_{1\leqslant n\leqslant m} n^{1/p} a_{l+n-1}(T) \tag{3.29}$$

for all l, m, $1 \leqslant l \leqslant m < \infty$.

THEOREM 3.4. *Let $0 < p < \infty$ and $T \in L(E, F)$. Then there exists a constant $C(p)$ such that*

$$\sup_{1\leqslant n\leqslant m} (l + n - 1)^{1/p}\, a_{n,l}(T) \leqslant C(p) \sup_{1\leqslant n\leqslant m} (l + n - 1)^{1/p}\, a_{l+n-1}(T) \tag{3.30}$$

for all l, m, $1 \leqslant l, m < \infty$.

Proofs. (Theorem 3.3) Suppose that n is given such that $1 \leqslant n \leqslant m$. We apply (3.23) (if E and F are complex Banach spaces cf. Remark 3.3) for l, m, n with $m = n$

$$n^{1/p} a_{n,l}(T) \leqslant n^{1/p} a_{l+n-1}(T) + 4 \log^2 4 \cdot C(p) \sup_{1 \leqslant k \leqslant n} k^{1/p} a_{l+k-1}(T).$$

For $\tilde{C}(p) = 1 + 4 \log^2 4 \cdot C(p)$ we obtain

$$\begin{aligned} n^{1/p} a_{n,l}(T) &\leqslant \tilde{C}(p) \sup_{1 \leqslant k \leqslant n} k^{1/p} a_{l+k-1}(T) \\ &\leqslant \tilde{C}(p) \sup_{1 \leqslant n \leqslant m} n^{1/p} a_{l+n-1}(T), \end{aligned}$$

that is, (3.29).

(Theorem 3.4) According to (3.29) we have for $l \leqslant m$

$$\begin{aligned} &\sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{n,l}(T) \\ &\quad \leqslant \sup_{1 \leqslant n \leqslant m} (2n)^{1/p} a_{n,l}(T) \leqslant 2^{1/p} C(p) \sup_{1 \leqslant n \leqslant m} n^{1/p} a_{l+n-1}(T) \\ &\quad \leqslant 2^{1/p} C(p) \sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{l+n-1}(T). \end{aligned} \tag{3.31}$$

Without any restrictions on l and m, we have

$$\begin{aligned} &\sup_{1 \leqslant n < l} (l+n-1)^{1/p} a_{n,l}(T) \\ &\quad \leqslant (2l)^{1/p} a_{1,l}(T) = (2l)^{1/p} a_l(T) \leqslant 2^{1/p} \sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{l+n-1}(T). \end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) we obtain (3.30). More precisely, for $l \leqslant m$ we use (3.31) and (3.32), and for $l > m$ we start from

$$\sup_{1 \leqslant n \leqslant m} (l+n-1)^{1/p} a_{n,l}(T) \leqslant \sup_{1 \leqslant n < l} (l+n-1)^{1/p} a_{n,l}(T)$$

and apply (3.32). ∎

The preceding two theorems are also valid for other s-covering numbers. We only mention the following result which corresponds to Theorem 3.4.

THEOREM 3.5. *Let $0<p<\infty$ and $T\in L(E,F)$. Then there exists a constant $C(p)$ such that*

$$\sup_{1\leqslant n\leqslant m}\ (l+n-1)^{1/p}\, a_{n,l}(T)\leqslant C(p)\sup_{1\leqslant n\leqslant m}\ (l+n-1)^{1/p}\, a_{l+n-1}(T) \tag{3.33}$$

for $s=c,d,t$ (cf. (2.2)–(2.4)) and all l,m, $1\leqslant l,m<\infty$.

Proof. For $s=t$ the desired estimate is an obvious consequence of Theorem 3.4 and the definition of t. For the other cases $s=c,d$ recall the definition of Gelfand and Kolmogorov numbers based on the approximation numbers

$$c_n(T)=a_n(I_\infty T),$$
$$d_n(T)=a_n(TQ_1),$$

where Q_1 and I_∞ are the canonical surjection and injection, respectively (cf. [CS]). In order to prove (3.33) for $s=c,d$ we show that

$$c_{n,l}(T)\leqslant 2a_{n,l}(I_\infty T) \tag{3.34}$$

$$d_{n,l}(T)\leqslant a_{n,l}(TQ_1). \tag{3.35}$$

These inequalities are consequences of

$$c_{n,l}(S)\leqslant a_{n,l}(S) \tag{3.36}$$

$$d_{n,l}(S)\leqslant a_{n,l}(S) \tag{3.37}$$

for any operator S. Since (3.37) is already proved in [CM] we only give the argument for the proof of (3.36). Let A be an operator with $\operatorname{rank}(A)<l$ such that

$$e_n(S-A)\leqslant a_{n,l}(S)+\varepsilon=\rho.$$

Hence

$$(S-A)(U_E)\subseteq\bigcup_{i=1}^{2^{n-1}}\{y_i+\rho U_F\}$$

for $y_i\in F$, $1\leqslant i\leqslant 2^{n-1}$. For $M=N(A)$, the null space of A, we have $\operatorname{codim}(M)<l$ and

$$(SI_M^E)(U_E)\subseteq\bigcup_{i=1}^{2^{n-1}}\{y_i+\rho U_F\},$$

hence

$$c_{n,l}(S)\leqslant e_n(SI_M^E)\leqslant a_{n,l}(S)+\varepsilon.$$

Because of the surjectivity (resp. injectivity) up to a factor 2 of the dyadic entropy numbers we have

$$c_{n,l}(T) \leqslant 2c_{n,l}(I_\infty T) \leqslant 2a_{n,l}(I_\infty T)$$
$$d_{n,l}(T) \leqslant d_{n,l}(TQ_1) \leqslant a_{n,l}(TQ_1).$$

These inequalities complete the proof of (3.33) for $s = c, d$. ▮

4. Inequalities between s-Covering Numbers, Gaussian Averages, and Rademacher Averages

This section is devoted to the investigation of inequalities between d-covering numbers of operators from a Hilbert space into a Banach space on one side and Gaussian or Rademacher averages on the other side. These inequalities complement and generalize V. Milman's discovering that the Gaussian average or the ℓ-norm is an appropriate parameter for estimating Gelfand and Kolmogorov numbers [M1, M2].

For this purpose recall the definition of the so-called Gaussian average or ℓ-norm of an operator T from ℓ_2^n into an arbitrary Banach space E. The ℓ-norm $\ell(T)$ of T is defined as

$$\ell(T) = \left(\int_{\mathbb{R}^n} \|Tx\|^2 \, d\gamma_n(x) \right)^{1/2}, \tag{4.1}$$

where γ_n denotes the canonical (normalized) Gaussian measure on the euclidean space $\mathbb{R}^n$. Moreover, for any operator T from ℓ_2 into E we define $\ell(T)$ as

$$\ell(T) = \sup\{\ell(TX) : X \in L(\ell_2^n, \ell_2) \text{ for some } n, \|X\| \leqslant 1\}. \tag{4.2}$$

We use a minoration of $\ell(T)$ which originated in the paper of Milman [M1] and was improved by A. Pajor and N. Tomczak-Jaegermann [PaT] (cf. [G, P2]) in the following theorem.

THEOREM 4.1. *Let E be a Banach space and let $T \in L(\ell_2, E)$ be a compact operator. Then*

$$\sup_{1 \leqslant k < \infty} k^{1/2} d_k(T) \leqslant C \cdot \ell(T), \tag{4.3}$$

where $C \geqslant 1$ is a universal constant.

Remark 4.1. Note that there is a dual version (4.3) for compact operators $T \in L(E, \ell_2)$, namely

$$\sup_{1 \leqslant k < \infty} k^{1/2} c_k(T) \leqslant C \cdot \ell(T'), \tag{4.4}$$

which is based upon $c_k(T)=d_k(T')$ for all operators T. But (4.3) is also a consequence of (4.4) because of $d_k(T)=c_k(T')$ for compact operators and $\ell(T)=\ell(T'')$, for $T\in L(\ell_2, E)$.

Combining (4.3) with Theorem 3.5 for $s=d$ we may state the following minoration theorem of the Gaussian average (ℓ-norm) by d-covering numbers.

THEOREM 4.2. *Let E be a Banach space and let $T\in L(\ell_2, E)$ be a compact operator. Then*

$$\sup_{1\leqslant l,n<\infty} (l+n-1)^{1/p}\, d_{n,l}(T)\leqslant C\cdot\ell(T), \tag{4.5}$$

where $C\geqslant 1$ is a universal constant.

The inequality (4.5) includes the classical Sudakov minoration theorem [Su] for $l=1$ and the inequality (4.3) for $n=1$. The version corresponding to (4.4) is the minoration of $\ell(T')$, $T\in L(E,\ell_2)$ by c-covering numbers:

$$\sup_{1\leqslant l,n<\infty} (l+n-1)^{1/p}\, c_{n,l}(T)\leqslant C\cdot\ell(T). \tag{4.6}$$

Next we want to derive similar inequalities for Rademacher averages instead of Gaussian averages. For this purpose let $\Phi=\{f_1, ..., f_m\}$ be an orthonormal basis of ℓ_2^m and $T\in L(\ell_2^m, E)$ be an operator. The Rademacher average or r-norm of T with respect to Φ is given by

$$r_\Phi(T)=\left(\operatorname*{Average}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{m}\varepsilon_i T(f_i)\right\|^2\right)^{1/2}.$$

It is well-known that $r_\Phi(T)\leqslant c\cdot\ell(T)$ for some universal constant c, independent of the special choice of Φ. The following minoration theorem for the Rademacher averages was proved in [CP].

THEOREM 4.3. *Let E be a Banach space and let $T\in L(\ell_2^m, E)$ be an operator of rank n. Then*

$$\sup_{1\leqslant k\leqslant n}\left(\log\left(1+\frac{n}{k}\right)\right)^{-1/2} k^{1/2}\, d_k(T)\leqslant C\cdot r_\Phi(T) \tag{4.7}$$

for any orthonormal basis Φ in ℓ_2^m. The constant C is a universal constant.

Combining Theorem 3.5 with Theorem 4.3, we obtain the Rademacher version of Theorem 4.2.

THEOREM 4.4. *Let E be a Banach space and let $T \in L(\ell_2^m, E)$ be an operator of* rank n. *Then*

$$\sup_{1 \leqslant k, l < \infty} \left(\log\left(1 + \frac{n}{k+l-1}\right)\right)^{-1/2} (k+l-1)^{1/2} d_{k,l}(T) \leqslant C \cdot \gamma_\Phi(T) \qquad (4.8)$$

for a universal constant C and any orthonormal basis Φ in ℓ_2^m.

Proof. Let $k \geqslant 1$. From (3.33) we get

$$\begin{aligned}
&(l+k-1)\, d_{k,l}(T) \\
&\quad \leqslant C(1) \sup_{1 \leqslant j \leqslant k} (l+j-1)\, d_{l+j-1}(T) \\
&\quad \leqslant C(1) \sup_{1 \leqslant j \leqslant k} (l+j-1)^{1/2} \left(\log\left(1 + \frac{n}{k+j-1}\right)\right)^{+1/2} \\
&\qquad \times \sup_{1 \leqslant j \leqslant k} (l+j-1)^{1/2} \left(\log\left(1 + \frac{n}{l+j-1}\right)\right)^{-1/2} d_{l+j-1}(T).
\end{aligned}$$

The first supremum on the right-hand side equals

$$(l+k-1)^{1/2} \left(\log\left(1 + \frac{n}{l+k-1}\right)\right)^{1/2}$$

since $x \to x \cdot \log(1 + n/x)$ is an increasing function. The second supremum can be estimated by $C \cdot \gamma_\Phi(T)$ according to (4.7). Hence

$$(l+k-1)^{1/2} \left(\log\left(1 + \frac{n}{l+k-1}\right)\right)^{-1/2} d_{k,l}(T) \leqslant C \cdot C(1) \cdot \gamma_\Phi(T). \quad \blacksquare$$

There are dual versions of (4.7) and (4.8) for operators $T \in L(E, \ell_2^m)$, of rank n, namely

$$\sup_{1 \leqslant k \leqslant n} \left(\log\left(1 + \frac{n}{k}\right)\right)^{-1/2} k^{1/2} c_k(T) \leqslant C \cdot \gamma_\Phi(T') \qquad (4.9)$$

and

$$\sup_{1 \leqslant l, k < \infty} \left(\log\left(1 + \frac{n}{k+l-1}\right)\right)^{-1/2} (k+l-1)^{1/2} c_{k,l}(T) \leqslant C \cdot \gamma_\Phi(T'). \quad (4.10)$$

The last inequality can be interpretated as follows:

Let $B \subset \mathbb{R}^n$ be a compact, convex, and symmetric set with non-empty interior. Equipped with the Minkowski functional $\|\cdot\|_B$ of B the vector space $\mathbb{R}^n$ becomes a normed vector space which is isomorphic to ℓ_2^n and admitting B as its unit ball. Let us apply (4.10) to the canonical isomorphism

$$T: (\mathbb{R}^n, \|\cdot\|_B) \to (\mathbb{R}^n, |\cdot|_2) = \ell_2^n.$$

COROLLARY 4.1. *For $0 < p < \infty$ there exists a constant C_p, only depending on p, and for each pair (k,l), $1 \leqslant k < \infty$, $1 \leqslant l < n$, there exists an l-dimensional subspace $M \subset \mathbb{R}^n$ such that*

$$e_k(M \cap B) \leqslant C_p \cdot \left\{ \underset{\varepsilon_i = \pm 1}{\text{Average}} \sup_{t = (t_1, \ldots, t_n) \in B} \left| \sum_{i=1}^{n} \varepsilon_i t_i \right|^p \right\}^{1/p} \times \left(\log \left(1 + \frac{n}{n+k-1} \right) \right)^{1/2} (n+k-1)^{1/2}. \tag{4.11}$$

Therefore $M \cap B$ is covered by 2^{k-1} euclidean balls of radius ρ, where ρ is bounded by the right-hand side of (4.11). *In particular, for each l, $1 \leqslant l \leqslant n$, there is an l-dimensional subspace $M \subset \mathbb{R}^n$ such that*

$$e_l(M \cap B) \leqslant C_p \cdot \left\{ \underset{\varepsilon_i = \pm 1}{\text{Average}} \sup_{t = (t_1, \ldots, t_n) \in B} \left| \sum_{i=1}^{n} \varepsilon_i t_i \right|^p \right\}^{1/p} n^{-1/2}. \tag{4.12}$$

For operators $T \in L(\ell_1^n, \ell_2)$ we have the following modification of (4.9), which was proved in [CP].

THEOREM 4.5. *There exists a universal constant C such that*

$$\sup_{1 \leqslant k \leqslant n} \left(\log \left(1 + \frac{n}{k} \right) \right)^{-1/2} k^{1/2} c_k(T) \leqslant C \, \|T\| \tag{4.13}$$

for all $T \in L(\ell_2^n, \ell_2)$ and all $n \in \mathbb{N}$.

As in Theorem 4.4, we derive from (4.13) and Theorem 3.5

$$\sup_{1 \leqslant k, l < \infty} \left(\log \left(1 + \frac{n}{k+l-1} \right) \right)^{-1/2} (k+l-1)^{1/2} c_{k,l}(T) \leqslant C \, \|T\| \tag{4.14}$$

for $T \in L(\ell_1^n, \ell_2)$.

If we take the volume ratio inequality

$$\left[\frac{\mathrm{vol}_m(A)}{\mathrm{vol}_m(U_{\ell_2^n})} \right]^{1/m} \leqslant \inf_{1 \leqslant k < \infty} k^{1/m} \varepsilon_k(A)$$

into consideration and denote the absolutely convex hull of the points $x_1, \ldots, x_n \in \ell_2^n$, i.e., the set

$$\left\{ \sum_{i=1}^{n} \lambda_i x_i \colon \sum_{i=1}^{n} |\lambda_i| \leqslant 1 \right\},$$

by $\mathrm{aconv}(x_1, \ldots, x_n)$, then we can derive the following corollary from (4.14).

COROLLARY 4.2. *Let* $x_1, ..., x_n \in \ell_2^n$ *be given, such that* $B = \text{aconv}(x_1, ..., x_n)$ *has a non-empty interior. For each* m, $1 \leqslant m < n$, *there exists an m-dimensional subspace* $M \subset \ell_2^n$ *with*

$$\left[\frac{\text{vol}_m(M \cap B)}{\text{vol}_m(U_{\ell_2^m})}\right]^{1/m} \leqslant C \cdot \left(\log\left(1 + \frac{n}{n-m+1}\right)\right)^{+1/2} \times (n-m+1)^{-1/2} \max_{1 \leqslant i \leqslant n} \|x_i\|_2,$$

where C is a universal constant.

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